

Lecture 2 (Predicates and Quantifiers)

Note that the sentence “ $P(x) := x + 2 = 2x$ ” is not a proposition. However, if we assign a value for x then it becomes a proposition. As for each value of x the sentence is either true or false. Thus the sentence can be treated as a function for which input is a value of x and the output is a proposition. Such sentence is an example of predicate or a propositional function. We define it more precise way as follows:

Predicate or Propositional function: Let A be a given set. A propositional function defined on A is an expression $P(x)$ which has the property that $P(a)$ is true or false for each $a \in A$. That is $P(x)$ becomes a statement whenever x is replaced by any value $a \in A$.

The set A is called the domain of $P(x)$, and the set T_p of all elements of A for which $P(a)$ is true is called the truth set of $P(x)$. In other words, $T_p = \{x : x \in A, P(x) \text{ is true}\}$

Example: Find the truth set T_p of each propositional function $P(x)$ defined on the set $\mathbb{N}$.

1. Let $P(x)$ be “ $x + 5 > 1$ ”. Then $T_p = \{x : x \in \mathbb{N}, x + 5 > 1\} = \mathbb{N}$.
2. Let $P(x)$ be “ $x + 2 > 7$ ”. Then $T_p = \{x : x \in \mathbb{N}, x + 2 > 7\} = \{6, 7, 8, \dots\}$ consists of all integers greater than 5.
3. Let $P(x)$ be “ $x + 5 < 3$ ”. Then $T_p = \{x : x \in \mathbb{N}, x + 5 < 3\} = \emptyset$.

Remark: The above example shows that if $P(x)$ is a propositional function defined on a set A then $P(x)$ could be true for all $x \in A$, for some $x \in A$ or for no $x \in A$. In the next paragraph, we discuss this quantifiers related notion to such proposition function.

A word which is usually used before noun to express the quantity of object is called quantifier. Here we discuss few quantifiers which are used in propositional functions.

Universal Quantifier:

Let $P(x)$ be a propositional function defined on a set A . Consider the expression

$$(\forall x \in A) P(x) \quad \text{or} \quad \forall x P(x)$$

which reads as “for every x in A , $P(x)$ is true statement. The symbol $\forall$ which reads “for all” or “for every” is called universal quantifier. In this case $T_p = A$ (the entire domain).

Existential Quantifier: Let $P(x)$ be a propositional function defined on a set A . Consider the expression

$$(\exists x \in A) P(x) \quad \text{or} \quad \exists x P(x)$$

which reads as “there exists x in A such that $P(x)$ is true statement. The symbol $\exists$ which reads “there exists” or “for some” or “for at least one” is called existential quantifier. In this case $T_p \neq \emptyset$.

Negation of Quantified Statements

Consider the statement “All maps are linear”. Its negation is either of the following equivalent statements:

“It is not the case that that all maps are linear”

“There exists at least one map which is not linear”.

Symbolically, let S denote the set of all maps. Then the above negation can be written as

$$\neg(\forall x \in S) (x \text{ is linear}) \equiv (\exists x \in S) (x \text{ is not linear}).$$

Or when $P(x)$ denotes “ x is linear”,

$$\neg(\forall x \in S) P(x) \equiv (\exists x \in S) \neg P(x) \quad \text{or} \quad \neg \forall x P(x) \equiv \exists x \neg P(x).$$

Thus we have:

1. $\neg(\forall x \in S) P(x) \equiv (\exists x \in S) \neg P(x)$
2. $\neg(\exists x \in S) P(x) \equiv (\forall x \in S) \neg P(x)$

The above rules for negations for quantifiers are called De Morgan’s laws for quantifiers.

Example: What are the negations of the statements $\forall x (x^2 > x)$ and $\exists x (x^2 = 2)$?

Solution: The negation of $\forall x (x^2 > x)$ is the statement $\neg \forall x (x^2 > x)$, which is equivalent to $\exists x \neg(x^2 > x)$, that is, $\exists x (x^2 \leq x)$. The negation of $\exists x (x^2 = 2)$ is the statement $\neg \exists x (x^2 = 2)$, which is equivalent to $\forall x \neg(x^2 = 2)$, that is, $\forall x (x^2 \neq 2)$.

Example: Show that $\neg \forall x (P(x) \rightarrow Q(x)) \equiv \exists x (P(x) \wedge \neg Q(x))$.

Solution: By De Morgan’s law for universal quantifiers, we know that $\neg \forall x (P(x) \rightarrow Q(x))$ and $\exists x (\neg(P(x) \rightarrow Q(x)))$ are logically equivalent. Since $P(x) \rightarrow Q(x) \equiv \neg P(x) \vee Q(x)$, it follows that $\neg \forall x (P(x) \rightarrow Q(x)) \equiv \exists x (P(x) \wedge \neg Q(x))$.

Nested Quantifiers: Two quantifiers are nested if one is within the scope of the other.

Example: The statement

$$\forall x \forall y (x + y = y + x)$$

says that $x + y = y + x$ for all real numbers x and y . This is the commutative law for addition of real numbers.

The statement

$$\forall x \exists y (x + y = 0)$$

says that for every real number x there is a real number y such that $x + y = 0$. This states that every real number has an additive inverse.

Similarly, the statement

$$\forall x \forall y \forall z (x + (y + z) = (x + y) + z)$$

is the associative law for addition of real numbers.

Quantifications of two variables

Statement	When True?	When False (or Negation of True)?
$\forall x \forall y P(x, y)$	$P(x, y)$ is true for every pair x, y	There is a pair x, y for which $P(x, y)$ is false
$\forall x \exists y P(x, y)$	For every x there is a y for which $P(x, y)$ is true	There is an x such that $P(x, y)$ is false for every y .
$\exists x \forall y P(x, y)$	There is an x for which $P(x, y)$ is true for every y .	For every x there is a y for which $P(x, y)$ is false.
$\exists x \exists y P(x, y)$	There is a pair x, y for which $P(x, y)$ is true.	$P(x, y)$ is false for every pair x, y .

Example: We can express that a function $f : X \rightarrow Y$ is one-to-one using quantifiers as

$$\forall a \forall b (f(a) = f(b) \rightarrow a = b).$$

Example: A function $f : X \rightarrow Y$ is onto if

$$\forall y \exists x (f(x) = y).$$

Thus f is not one-to-one if there is a pair a, b for which $f(a) = f(b) \rightarrow a \neq b$.

Example: Use quantifiers to express the definition of the limit of a real-valued function $f(x)$ of a real variable x at a point a in its domain.

Solution: Recall that the definition of the statement

$$\lim_{x \rightarrow a} f(x) = L$$

is: For every real number $\epsilon > 0$ there exists a real number $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $0 < |x - a| < \delta$. This definition of a limit can be phrased in terms of quantifiers by

$$\exists L \in \mathbb{R} \forall \epsilon > 0 \exists \delta > 0 \forall x \left(0 < |x - a| < \delta \rightarrow |f(x) - L| < \epsilon \right).$$

Thus negation of above is:

$$\forall L \in \mathbb{R} \exists \epsilon > 0 \forall \delta > 0 \exists x \left(0 < |x - a| < \delta \wedge |f(x) - L| \geq \epsilon \right).$$

Example from “A basic course in Real Analysis by S Kumaresan”: Suppose we have a sentence: “In each tree in the orchard, we can find a branch in which all the leaves are green”.

Let us convert the above sentence as a mathematical sentence: Let T denote the set of all trees in the orchard. Let $t \in T$ be a tree. Let B_t denote the set of all branches of the tree t . Let $b \in B_t$ be a branch of tree t . Let L_b denote the set of all leaves on the branch b . Then the above sentence can be written as:

$\forall t \in T \exists b \in B_t \forall l \in L_b, l$ is green. The negation is:

$$\neg(\forall t \in T \exists b \in B_t \forall l \in L_b, l \text{ is green}) \equiv \exists t \in T \forall b \in B_t \exists l \in L_b, l \text{ is not green.}$$